

WIMP Dark Matter and the QCD Equation of State

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Weakly Interacting Massive Particles (WIMPs) of mass m freeze out at a temperature $T_f \simeq m/25$, i.e. in the range 400 MeV – 40 GeV for a particle in the typical mass range 10 – 1000 GeV. The WIMP relic density, which depends on the effective number of relativistic degrees of freedom at T_f , may be measured to better than 1% by Planck, warranting comparable theoretical precision. Recent theoretical and experimental advances in the understanding of high temperature QCD show that the quark gluon plasma departs significantly from ideal behaviour up to temperatures of several GeV, necessitating an improvement of the cosmological equation of state over those currently used. We discuss how this increases the relic density by approximately 1.5 – 3.5% in benchmark mSUGRA models, with an uncertainty in the QCD corrections of 0.5 – 1 %. We point out what further work is required to achieve a theoretical accuracy comparable with the expected observational precision, and speculate that the effective number of degrees of freedom at T_f may become measurable in the foreseeable future.

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A leading candidate for the primary constituent of dark matter is the Weakly Interacting Massive Particle (WIMP, see e.g. [1]). WIMPs of mass m generically freeze out at a temperature of around $T_f \simeq m/25$ GeV, and so for typical masses in the range 10 – 1000 GeV, T_f lies between 400 MeV and 40 GeV. The equation governing the relic density depends on both energy and entropy densities, and so the WIMP relic density is sensitive to the equation of state of the Universe in this temperature range. It is normally supposed that, above the QCD confinement critical temperature of $T_c \sim 200$ MeV, the plasma is weakly interacting because of asymptotic freedom, and can be treated as an ideal gas, and this assumption is built into at least two of the best-known SUSY WIMP relic density packages, DarkSUSY [2] and MicrOMEGAs [3].

However, intensive non-perturbative studies and experiments revealed that the high-temperature QCD plasma still departs significantly from an ideal gas at temperatures several orders of magnitude higher than T_c . It is therefore worth re-examining the equation of state for the Universe in this critical regime, and qualitative investigations have been made before [4]. Using recent progress in both the deconfined [5] and confined [6] phases, we construct an improved equation of state and investigate the consequences for the WIMP relic density in one of the dark matter packages, DarkSUSY. DarkSUSY currently uses an equation of state [7] based on work in Refs. [8, 9]. Replacing it with the improved version gives upward corrections of over 2% to the relic densities in a set of benchmark mSUGRA models [10], and larger effects for models with lower T_f .

Although small, this correction is becoming significant in the new era of precision cosmology: the cold dark mat-

ter (CDM) density is determined to better than 10% in single-field inflation models from a combination of the Cosmic Microwave Background (CMB) angular power spectrum and observations of the galaxy power spectrum by 2dF [11] and by SDSS [12]. Planck promises to do much better, with one estimate giving a determination to better than 1% [13].

The calculations we use still involve simplifications and systematic errors, forcing us to model $p(T)$ to some extent, as described below. Estimating our systematic errors on the QCD equation of state to be on the order of 10% near T_c , we show how they propagate into uncertainties in the range of 0.5 – 1% in the relic density of WIMPs. Thus, the quest for new physics in observational cosmology depends on further improvements in the accuracy of the quantitative understanding of the QCD equation of state.

There has been much effort in calculating the pressure $p(T)$ of an $SU(N_c)$ gauge theory with N_f fundamental fermions at temperature T , as reviewed in Refs. [14, 15, 16, 17]. Perturbative expansions in the coupling constant g of quantities such as the pressure converge badly, and particularly for a strongly-coupled theory like QCD. Strictly perturbative expansions, even when expanded to $O(g^6 \ln g)$ [5], seem to converge well only at remarkably high temperatures, above $10^3 T_c$, in sharp contradiction to the ideal gas assumption usually made in cosmology.

In the high temperature regime, progress has been made using perturbative finite-temperature dimensional reduction (DR) (see [5] and references therein). By constructing a sequence of effective theories for the scales $2\pi T$, gT and $g^2 T$, the last of which has to be treated non-perturbatively, one can get results for arbitrary N_f

of massless quarks applicable down to a few times T_c . By fitting for the non-perturbative and as yet unknown $O(g^6)$ coefficient, the authors of Ref. [5] were able to match their calculated pressure reasonably well to pure-glue lattice data near the critical temperature.

Around the transition, there now exist lattice calculations for the pressure and energy density for $N_f = 0, 2, 3$ degenerate quark flavours, as well as first data for $N_f = 2 + 1$, i.e. two light and one heavier flavour. The pseudo-critical temperatures $T_c(N_f)$ (defined as the peak of a susceptibility) are currently given as $T_c(0) = 271 \pm 2$ MeV, $T_c(2) = 173 \pm 8$ MeV and $T_c(3) = 154 \pm 8$ MeV [18]. It has to be stressed, however, that only the pure glue case has been extrapolated to the continuum. Based on experience with that theory, the dynamical fermion results are estimated to display a systematic error of about 15% at the currently available lattice spacing. Corrections due to quark masses deviating from the physical ones appear to be negligible in comparison [18].

Finally, below the phase transition the hadronic resonance gas model, which treats the plasma as an ideal gas of mesons, baryons and their excited states, matches reasonably well to lattice data [6].

We now review relic density calculations. Consider a particle of mass m and number density n , undergoing annihilations $XX \rightarrow \dots$ with total cross-section σ , assumed to be a typical weak interaction cross-section, proportional to G_F^2 . Then [19]

$$\dot{n} + 3Hn = -\langle\sigma v_{\text{Møl}}\rangle (n^2 - n_{\text{eq}}^2), \quad (1)$$

where n_{eq} is the equilibrium number density, $H = \dot{a}/a$ is the Hubble parameter, and $v_{\text{Møl}}$ is the Møller velocity which is a relativistic generalisation of the relative velocity of the annihilating particles [19].

In order to solve the equation it is convenient to convert the time variable to $x = m/T$, and to measure the relic abundance in terms of $Y = n/s$, where s is the entropy density. If the total entropy $S = sa^3$ is conserved, then we can write

$$\frac{dY}{dx} = \frac{1}{3H} \frac{ds}{dx} \langle\sigma v_{\text{Møl}}\rangle (Y^2 - Y_{\text{eq}}^2). \quad (2)$$

This adiabaticity assumption is violated if the QCD transition is first order, but it is most likely to be a cross-over transition at the low chemical potentials which are relevant for the early Universe [16, 17].

Using the Friedmann equation $H^2 = 8\pi G\rho/3$, and defining effective numbers of degrees of freedom for the energy and entropy densities, $s = (\rho + p)/T$, through

$$\rho = \frac{\pi^2}{30} g_{\text{eff}}(T) T^4, \quad s = \frac{2\pi^2}{45} h_{\text{eff}}(T) T^3, \quad (3)$$

one finds an approximate solution

$$Y_0 \simeq \left(\frac{45}{\pi}\right)^{\frac{1}{2}} \frac{1}{m M_{\text{P}} \langle\sigma v_{\text{Møl}}\rangle_{T_f}} \frac{x_f}{g_*^{\frac{1}{2}}(T_f)}, \quad (4)$$

where T_f is the freeze-out temperature, defined to be the temperature at which the relic abundance is a certain factor (taken to be 2.5) above the equilibrium abundance. We see that the relic density depends on the parameter

$$g_*^{1/2}(T) = \frac{h_{\text{eff}}}{g_{\text{eff}}^{1/2}} \left(1 + \frac{T}{3} \frac{d \ln h_{\text{eff}}}{dT}\right). \quad (5)$$

It is through this parameter that the QCD equation of state influences the relic density.

In an ideal gas at temperature T , a particle of mass $m_i = x_i T$ contributes to g_{eff} , h_{eff} the amounts

$$g_{i,\text{eff}} \equiv \frac{\rho_i}{\rho_0} = \frac{15}{\pi^4} \int_{x_i}^{\infty} \frac{(u^2 - x_i^2)^{\frac{1}{2}}}{e^u \pm 1} u^2 du, \quad (6)$$

$$h_{i,\text{eff}} \equiv \frac{s_i}{s_0} = \frac{45}{12\pi^4} \int_{x_i}^{\infty} \frac{(u^2 - x_i^2)^{\frac{1}{2}}}{e^u \pm 1} (4u^2 - x_i^2) du, \quad (7)$$

where ρ_0 and s_0 are the energy and entropy densities for a free massless boson.

Interactions correct the ideal gas result, and g_{eff} , h_{eff} have to be extracted from calculations of the energy and entropy, Eqs. (3). Note that in the relevant temperature range 40 to 0.4 GeV, 16 out of the 18 bosonic degrees of freedom are coloured, with the coloured fermionic degrees of freedom dropping from 60/78 to 36/50. The dominant corrections are therefore expected to come from the coloured sector of the Standard Model, weak corrections are moreover suppressed by the boson masses and negligible.

The relic density codes DarkSUSY and MicrOMEGAs use identical equations of state, developed in Refs. [7, 8, 9]. Below T_c the hadronic degrees of freedom are modelled by an interacting gas of hadrons and their resonances, while above T_c the quarks and gluons are taken to interact with a linear potential $V_Q(r) = Kr$, with a phenomenologically motivated value $K = 0.18 \text{ GeV}^2$, derived from the slope of Regge trajectories. In this model, the pressure is already very close to ideal at temperatures above 1.6 GeV. All other Standard Model particles are free.

In this work we also take the ideal gas contributions for the particles of the Standard Model, with masses given by the Particle Data Group central values [20]. In the confined phase quarks and gluons are replaced by hadronic models described below. In the deconfined phase, the contribution to the pressure of the coloured degrees of freedom is scaled by a function $f(T)$, defined to be the ratio between the true QCD pressure $p(T)$ and the Stefan-Boltzmann result p_{SB} for the same theory, $f(T) = p(T)/p_{\text{SB}}(T)$. This correction factor is derived from lattice [18] and perturbative [5] calculations for $N_f = 0$, and uses an approximate universality in the pressure curves for different N_f observed by Karsch et al. [18]. Near the transition, the lattice-derived curves for $f(T)$ have the approximate form $f(T, N_f) = f_{\text{QCD}}(T/T_c(N_f))$,

where $T_c(N_f)$ is the critical temperature for the theory with N_f light fermion flavours. Besides this, there appears to be only negligible N_f -dependence within the current numerical accuracy. We are therefore motivated to neglect quark mass effects, take the $N_f = 0$ lattice data (in the continuum limit) and the $N_f = 0$ DR formula of Kajantie et al. [5], and scale the temperature dependence by $T_c(3)/T_c(0)$. The correction factors are matched at 1.2 GeV using the undetermined $O(g^6)$ parameter, which we take to be 0.6755, close to the value 0.7 used in Ref. [5]. At higher temperatures one crosses the c and b mass thresholds. However, for $N_f > 3$ the $O(g^6)$ fitting parameter is unknown, and the appropriate critical temperatures are also unknown. Hence we believe that scaling the $N_f = 0$ result is the best we can do at present. We will discuss later how improvements can be made.

In the confined phase we label our model equations of state (EOS) A,B and C. EOS A ignores hadrons completely, as the lattice shows that $f(T)$ very rapidly approaches zero below T_c . EOS B and C model hadrons as a gas of free mesons and baryons. It was noted in Ref. [6] that such a gas, including all resonances, gives a pressure which fits remarkably well to the $N_f = 2 + 1$ lattice results although, as the authors themselves point out, this result should be treated with caution as the simulations are not at the continuum limit. We include all resonances listed in the Particle Data Group's table `mass_width_02.mc` [20].

We make a sharp switch to the hadronic gas at a temperature T_{HG} . For our EOS B we take $T_{\text{HG}} = T_c = 154$ MeV, and for EOS C we take $T_{\text{HG}} = 200$ MeV and $T_c = 185.5$ MeV, values chosen to give as smooth a curve for h_{eff} as possible. The effects of these equations of state on the relic densities turn out to differ by less than 0.3% in the relevant temperature interval, so in the following we concentrate on EOS B.

Before presenting our results we note that $\Omega_c h^2$ is directly proportional to the entropy density today $s_0 = (2\pi^2/45)h_{\text{eff}}(T_\gamma)T_\gamma^3$, and that this must be determined as accurately as possible. The photon temperature $T_\gamma = 2.725 \pm 0.001$ is very accurately measured [21], but the contribution from neutrinos requires a separate freeze-out calculation. Recent work [22] gives $h_{\text{eff}}(T_\gamma) = 3.9172$. We find that taking freeze-out temperatures to be 3.5 MeV for ν_μ and ν_τ and 2 MeV for ν_e , as recommended in Ref. [7], gives $h_{\text{eff}}(T_\gamma) = 3.9138$, which is accurate enough for our purposes. DarkSUSY uses $h_{\text{eff}}(T_\gamma) = 3.9139$.

In Figs. 1 we plot for our EOS B the effective numbers of degrees of freedom $h_{\text{eff}}(T)$ and $g_*^{1/2}(T)$ defined in Eqs. (3) and (5), compared with those used in DarkSUSY [2] and MicrOMEGAs [3]. The spike in our $g_*^{1/2}(T)$ is an artefact of the matching of the scaled lattice data and the hadronic equation of state (i.e. the first derivative jumps). Since for physical QCD the transition is a

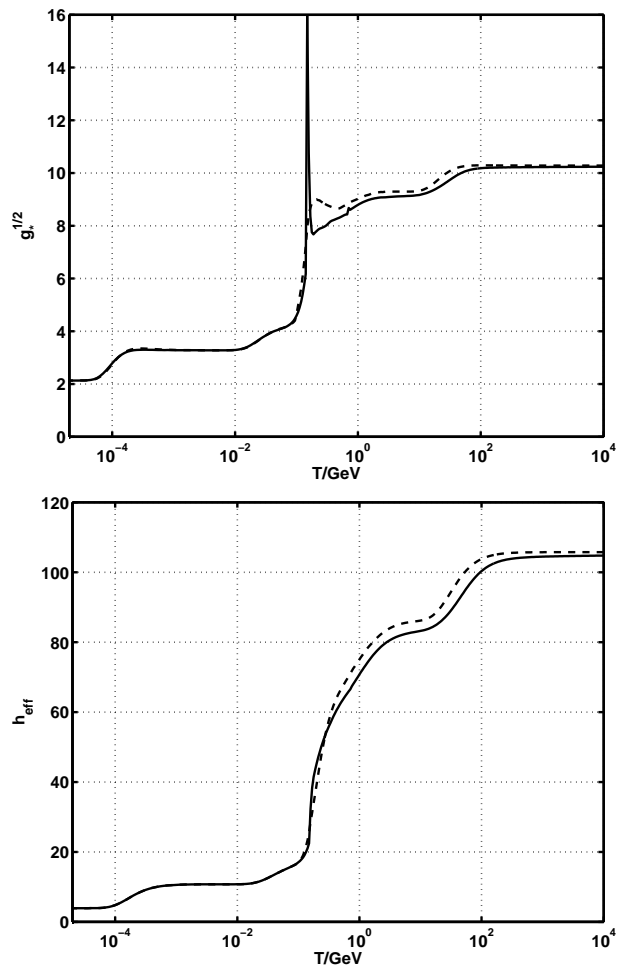


FIG. 1: Degrees of freedom factors $g_*^{1/2}(T)$ and $h_{\text{eff}}(T)$, defined in Eqs. 3 and 5, for our equation of state B (solid) compared with those currently used in DarkSUSY and MicrOMEGAs (dashed). The spike in the upper panel is a numerical artefact without physical impact, see text.

smooth crossover, this spike is unphysical. However, it has no noticeable influence on the freeze-out of WIMPs at higher temperatures.

In Table I we exhibit the effect of the new equation of state on the density of relic neutralinos χ , for the mSUGRA models used to test DarkSUSY in the standard distribution [23]. We find changes of about 1.5–3.5%. In order to quantify the effect of uncertainty in the lattice data, we introduce two new models B2 and B3, which are constructed by scaling the lattice curve by 0.9 and 1.1 respectively, and then adjusting the $O(g^6)$ parameter in the DR pressure curve so that it meets the scaled lattice curve at $T = 4.43T_c$. Thus a 10% uncertainty in the lattice pressure curve translates to an uncertainty in the relic density in the range 0.5 – 1%. Note that the lowest freeze-out temperature in the table is about 4 GeV. This corresponds to more than $20T_c$, where the

Model	m_χ/GeV	T_f/GeV	$\Omega_c h^2$ (DS)	$\Omega_c h^2$ (here)	$\Delta(\%)$
A'	242.83	9.8	0.0929	0.0948 ⁽⁵⁴⁾ ₍₄₂₎	2.0 ^{2.6} _{1.4}
B'	94.88	4.1	0.1213	0.1242 ⁽⁵⁶⁾ ₍₃₁₎	2.4 ^{3.6} _{1.5}
C'	158.09	6.5	0.1149	0.1174 ⁽⁸³⁾ ₍₆₅₎	2.2 ^{2.9} _{1.5}
D'	212.42	8.6	0.0864	0.0882 ⁽⁸⁸⁾ ₍₇₆₎	2.0 ^{2.7} _{1.4}
G'	147.98	6.2	0.1294	0.1323 ⁽³³⁾ ₍₁₃₎	2.2 ^{3.0} _{1.4}
H'	388.38	16.0	0.1629	0.1662 ⁽⁷¹⁾ ₍₅₃₎	2.0 ^{2.6} _{1.5}
I'	138.08	5.8	0.1319	0.1351 ⁽⁶²⁾ ₍₄₀₎	2.4 ^{3.2} _{1.6}
J'	309.17	12.6	0.0966	0.0984 ⁽⁹⁰⁾ ₍₇₉₎	2.0 ^{2.5} _{1.4}
K'	554.19	22.9	0.0863	0.0883 ⁽⁸⁸⁾ ₍₇₈₎	2.3 ^{3.0} _{1.8}
L'	180.99	7.5	0.0988	0.1011 ⁽¹⁸⁾ ₍₀₃₎	2.3 ^{3.0} _{1.5}

TABLE I: Relic densities in benchmark models [10], calculated with DarkSUSY. Displayed: neutralino mass m_χ , freeze-out temperature T_f , $\Omega_c h^2$ computed using the current DarkSUSY equation of state, $\Omega_c h^2$ using our QCD-corrected Equations of State $B_{(B3)}^{(B2)}$, and percentage changes Δ .

QCD corrections to g_* are around 5%. Evidently, WIMPs with freeze-out temperatures closer to T_c , such as sterile neutrinos [4], would be affected more strongly.

To conclude: by updating the equation of state of the Standard Model in the light of recent developments in high temperature QCD, we have found differences in WIMP relic density calculations of a few per cent for the benchmark models of Ref. [10], a small but not insignificant effect. These models freeze out well above T_c , and so it is likely that there are other models exhibiting greater changes. In single field inflation models, the 68% confidence limit on the dark matter density is already $\Omega_c h^2 = 0.127 \pm 0.017$ from the WMAP First Year data [11], while Planck has been estimated to be able to reach a level of $\Delta\Omega_c h^2 = 0.0011$ [13]. In order to benefit from this accuracy, it has to be matched by that of theoretical calculations of the QCD equation of state. To obtain an accuracy of 1% we estimate that we require lattice data accurate to 10% at about $T \simeq 4T_c$, for realistic quark masses, and in the continuum limit. Further progress is possible: the $O(g^6)$ term is in principle calculable but requires a 4-loop computation in lattice perturbation theory [5]. Moreover, the b and c quark mass thresholds may affect the QCD corrections. We also note that the equation of state at a few T_c is also experimentally accessible in current and future heavy ion experiments, which gives them a direct cosmological motivation.

Finally, given a sufficiently accurate g_* , we speculate that with a combination of measurements of the neutralino mass and cross-sections in direct detection experiments and at LHC and the ILC, the effective number of degrees of freedom in the early Universe at a few GeV will become experimentally accessible, motivating attempts to improve further on this work.

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